

A note on the Dancer-Fučík spectra of the fractional p -Laplacian and Laplacian operators*

Kanishka Perera

Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, FL 32901, USA

Marco Squassina[†]

Dipartimento di Informatica
Università degli Studi di Verona
37134 Verona, Italy

Yang Yang[‡]

School of Science
Jiangnan University
Wuxi, 214122, China

Abstract

We study the Dancer-Fučík spectrum of the fractional p -Laplacian operator. We construct an unbounded sequence of decreasing curves in the spectrum using a suitable minimax scheme. For $p = 2$, we present a very accurate local analysis. We construct the minimal and maximal curves of the spectrum locally near the points where it intersects the main diagonal of the plane. We give a sufficient condition for the region between them to be nonempty, and show that it is free of the spectrum in the case of a simple eigenvalue. Finally we compute the critical groups in various regions separated by these curves. We compute them precisely in certain regions, and prove a shifting theorem that gives a finite-dimensional reduction in certain other regions. This allows us to obtain nontrivial solutions of perturbed problems with nonlinearities crossing a curve of the spectrum via a comparison of the critical groups at zero and infinity.

1 Introduction

For $p \in (1, \infty)$, $s \in (0, 1)$, and $N > sp$, the fractional p -Laplacian $(-\Delta)_p^s$ is the nonlinear nonlocal operator defined on smooth functions by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+sp}} dy, \quad x \in \mathbb{R}^N.$$

**MSC2010*: Primary 35R11, 35P30, Secondary 35A15

Key Words and Phrases: fractional p -Laplacian, Dancer-Fučík spectrum, critical groups

[†]The second-named author was supported by 2009 MIUR project: “Variational and Topological Methods in the Study of Nonlinear Phenomena”.

[‡]This work was completed while the third-named author was visiting the Department of Mathematical Sciences at the Florida Institute of Technology, and she is grateful for the kind hospitality of the department. Project supported by NSFC-Tian Yuan Special Foundation (No. 11226116), Natural Science Foundation of Jiangsu Province of China for Young Scholars (No. BK2012109), and the China Scholarship Council (No. 201208320435).

This definition is consistent, up to a normalization constant depending on N and s , with the usual definition of the linear fractional Laplacian operator $(-\Delta)^s$ when $p = 2$. There is currently a rapidly growing literature on problems involving these nonlocal operators. In particular, fractional p -eigenvalue problems have been studied in Lindgren and Lindqvist [29], Iannizzotto and Squassina [25], and Franzina and Palatucci [20], regularity of fractional p -minimizers in Di Castro et al. [15], and existence via Morse theory in Iannizzotto et al. [24]. We refer to Caffarelli [6] for the motivations that have lead to their study.

Let Ω be a bounded domain in \mathbb{R}^N with Lipschitz boundary $\partial\Omega$. The Dancer-Fučík spectrum of the operator $(-\Delta)_p^s$ in Ω is the set $\Sigma_p^s(\Omega)$ of all points $(a, b) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} (-\Delta)_p^s u = b(u^+)^{p-1} - a(u^-)^{p-1} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.1)$$

where $u^\pm = \max\{\pm u, 0\}$ are the positive and negative parts of u , respectively, has a nontrivial weak solution. Let us recall the weak formulation of (1.1). Let

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p}$$

be the Gagliardo seminorm of the measurable function $u : \mathbb{R}^N \rightarrow \mathbb{R}$, and let

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\}$$

be the fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} = (|u|_p^p + [u]_{s,p}^p)^{1/p},$$

where $|\cdot|_p$ is the norm in $L^p(\mathbb{R}^N)$. We work in the closed linear subspace

$$X_p^s(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,p}$ (see Di Nezza et al. [16, Theorem 7.1]). A function $u \in X_p^s(\Omega)$ is a weak solution of problem (1.1) if

$$\begin{aligned} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+sp}} dx dy \\ = \int_{\Omega} [b(u^+)^{p-1} - a(u^-)^{p-1}] v dx \quad \forall v \in X_p^s(\Omega). \end{aligned} \quad (1.2)$$

This notion of spectrum for linear local elliptic partial differential operators was introduced by Dancer [10, 11] and Fučík [21], who recognized its significance for the solvability of related semilinear boundary value problems. In particular, the Dancer-Fučík spectrum of the Laplacian in Ω with the Dirichlet boundary condition is the set $\Sigma(\Omega)$ of all points $(a, b) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} -\Delta u = bu^+ - au^- & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.3)$$

has a nontrivial solution. Denoting by $\lambda_k \nearrow +\infty$ the Dirichlet eigenvalues of $-\Delta$ in Ω , $\Sigma(\Omega)$ clearly contains the sequence of points (λ_k, λ_k) . For $N = 1$, where Ω is an interval, Fučík [21] showed that $\Sigma(\Omega)$ with the periodic boundary condition consists of a sequence of hyperbolic-like curves passing through the points (λ_k, λ_k) , with one or two curves going through each point. For $N \geq 2$, $\Sigma(\Omega)$ consists locally of

curves emanating from the points (λ_k, λ_k) (see Gallouët and Kavian [22], Ruf [42], Lazer and McKenna [26], Lazer [27], Căc [5], Magalhães [32], Cuesta and Gossez [9], de Figueiredo and Gossez [14], and Margulies and Margulies [33]). Schechter [43] showed that in the square $(\lambda_{k-1}, \lambda_{k+1}) \times (\lambda_{k-1}, \lambda_{k+1})$, $\Sigma(\Omega)$ contains two strictly decreasing curves, which may coincide, such that the points in the square that are either below the lower curve or above the upper curve are not in $\Sigma(\Omega)$, while the points between them may or may not belong to $\Sigma(\Omega)$ when they do not coincide.

The Dancer-Fučík spectrum of the p -Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the set $\Sigma_p(\Omega)$ of all points $(a, b) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} -\Delta_p u = b(u^+)^{p-1} - a(u^-)^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a nontrivial solution. For $N = 1$, the Dirichlet spectrum $\sigma(-\Delta_p)$ of $-\Delta_p$ in Ω consists of a sequence of simple eigenvalues $\lambda_k \nearrow +\infty$ and $\Sigma_p(\Omega)$ has the same general shape as $\Sigma(\Omega)$ (see Drábek [17]). For $N \geq 2$, the first eigenvalue λ_1 of $-\Delta_p$ is positive, simple, and has an associated eigenfunction that is positive in Ω (see Anane [4] and Lindqvist [30, 31]), so $\Sigma_p(\Omega)$ contains the two lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$. Moreover, λ_1 is isolated in the spectrum, so the second eigenvalue $\lambda_2 = \inf \sigma(-\Delta_p) \cap (\lambda_1, \infty)$ is well-defined (see Anane and Tsouli [2]), and a first nontrivial curve in $\Sigma_p(\Omega)$ passing through (λ_2, λ_2) and asymptotic to $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ at infinity was constructed using the mountain pass theorem by Cuesta et al. [8]. Although a complete description of $\sigma(-\Delta_p)$ is not yet available, an increasing and unbounded sequence of eigenvalues can be constructed via a standard minimax scheme based on the Krasnosel'skiĭ genus, or via nonstandard schemes based on the cogenus as in Drábek and Robinson [18] and the cohomological index as in Perera [35]. Unbounded sequences of decreasing curves in $\Sigma_p(\Omega)$, analogous to the lower and upper curves of Schechter [43] in the semilinear case, have been constructed using various minimax schemes by Cuesta [7], Micheletti and Pistoia [34], and Perera [36].

Goyal and Sreenadh [23] recently studied the Dancer-Fučík spectrum for a class of linear nonlocal elliptic operators that includes the fractional Laplacian $(-\Delta)^s$. As in Cuesta et al. [8], they constructed a first nontrivial curve in the Dancer-Fučík spectrum that passes through (λ_2, λ_2) and is asymptotic to $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ at infinity. Very recently, in [3], the authors proved, among other things, that the second variational eigenvalue λ_2 is larger than λ_1 and (λ_1, λ_2) does not contain other eigenvalues.

The purpose of this note is to point out that the general theories developed in Perera et al. [37] and Perera and Schechter [41] apply to the fractional p -Laplacian and Laplacian operators, respectively, and draw some conclusions about their Dancer-Fučík spectra. We construct an unbounded sequence of decreasing curves in $\Sigma_p^s(\Omega)$ using a suitable minimax scheme. For $p = 2$, we present a very accurate local analysis. We construct the minimal and maximal curves of the spectrum locally near the points where it intersects the main diagonal of the plane. We give a sufficient condition for the region between them to be nonempty, and show that it is free of the spectrum in the case of a simple eigenvalue. Finally we compute the critical groups in various regions separated by these curves. We compute them precisely in certain regions, and prove a shifting theorem that gives a finite-dimensional reduction in certain other regions. This allows us to obtain nontrivial solutions of perturbed problems with nonlinearities crossing a curve of the spectrum via a comparison of the critical groups at zero and infinity.

2 Dancer-Fučík spectrum of the fractional p -Laplacian

The general theory developed in Perera et al. [37] applies to problem (1.1). Indeed, the odd $(p-1)$ -homogeneous operator $A_p^s \in C(X_p^s(\Omega), X_p^s(\Omega)^*)$, where $X_p^s(\Omega)^*$ is the dual of $X_p^s(\Omega)$, defined by

$$A_p^s(u) v = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+sp}} dx dy, \quad u, v \in X_p^s(\Omega)$$

that is associated with the left-hand side of equation (1.2) satisfies

$$A_p^s(u) u = \|u\|^p, \quad |A_p^s(u) v| \leq \|u\|^{p-1} \|v\| \quad \forall u, v \in X_p^s(\Omega) \quad (2.1)$$

and is the Fréchet derivative of the C^1 -functional

$$I_p^s(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy, \quad u \in X_p^s(\Omega).$$

Moreover, since $X_p^s(\Omega)$ is uniformly convex, it follows from (2.1) that A_p^s is of type (S), i.e., every sequence $(u_j) \subset X_p^s(\Omega)$ such that

$$u_j \rightharpoonup u, \quad A_p^s(u_j)(u_j - u) \rightarrow 0$$

has a subsequence that converges strongly to u (see [37, Proposition 1.3]). Hence the operator A_p^s satisfies the structural assumptions of [37, Chapter 1].

When $a = b = \lambda$, (1.1) reduces to the nonlinear eigenvalue problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (2.2)$$

Eigenvalues of this problem coincide with critical values of the functional

$$\Psi(u) = \left(\int_{\Omega} |u|^p dx \right)^{-1}$$

on the manifold

$$\mathcal{M} = \{u \in X_p^s(\Omega) : \|u\| = 1\}.$$

The first eigenvalue

$$\lambda_1 = \inf_{u \in \mathcal{M}} \Psi(u)$$

is positive, simple, isolated, and has an associated eigenfunction that is positive in Ω (see Lindgren and Lindqvist [29] and Franzina and Palatucci [20]), so $\Sigma_p^s(\Omega)$ contains the two lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$. Let \mathcal{F} denote the class of symmetric subsets of \mathcal{M} , let $i(M)$ denote the \mathbb{Z}_2 -cohomological index of $M \in \mathcal{F}$ (see Fadell and Rabinowitz [19]), and set

$$\lambda_k := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \Psi(u), \quad k \geq 2.$$

Then $\lambda_k \nearrow +\infty$ is a sequence of eigenvalues of problem (2.2) (see [37, Theorem 4.6]), so $\Sigma_p^s(\Omega)$ contains the sequence of points (λ_k, λ_k) .

Following [37, Chapter 8], we now construct an unbounded sequence of decreasing curves in $\Sigma_p^s(\Omega)$. For $t > 0$, let

$$\Psi_t(u) = \left(\int_{\Omega} [(u^+)^p + t(u^-)^p] dx \right)^{-1}, \quad u \in \mathcal{M}.$$

Then the point $(c, ct) \in \Sigma_p^s(\Omega)$ if and only if c is a critical value of Ψ_t (see [37, Lemma 8.3]). For each $k \geq 2$ such that $\lambda_k > \lambda_{k-1}$, let

$$C\Psi_t^{\lambda_{k-1}} = (\Psi_t^{\lambda_{k-1}} \times [0, 1]) / (\Psi_t^{\lambda_{k-1}} \times \{1\})$$

be the cone on the sublevel set $\Psi_t^{\lambda_{k-1}} = \{u \in \mathcal{M} : \Psi_t(u) \leq \lambda_{k-1}\}$, let Γ_k denote the class of maps $\gamma \in C(C\Psi_t^{\lambda_{k-1}}, \mathcal{M})$ such that $\gamma|_{\Psi_t^{\lambda_{k-1}}}$ is the identity, and set

$$c_k^s(t) = \inf_{\gamma \in \Gamma_k} \sup_{u \in \gamma(C\Psi_t^{\lambda_{k-1}})} \Psi_t(u).$$

We have the following theorem as a special case of [37, Theorem 8.8].

Theorem 2.1. *Let*

$$\mathcal{C}_k^s = \{(c_k^s(t), c_k^s(t)t) : \lambda_{k-1}/\lambda_k < t < \lambda_k/\lambda_{k-1}\}.$$

Then \mathcal{C}_k is a decreasing continuous curve in $\Sigma_p^s(\Omega)$, and $c_k^s(1) \geq \lambda_k$.

3 Dancer-Fučík spectrum of the fractional Laplacian

The Dancer-Fučík spectrum of the operator $(-\Delta)^s$ in Ω is the set $\Sigma^s(\Omega)$ of all points $(a, b) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} (-\Delta)^s u = bu^+ - au^- & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \quad (3.1)$$

has a nontrivial weak solution. The general theory developed in Perera and Schechter [41] applies to problem (3.1). Indeed, set $X^s(\Omega) = X_2^s(\Omega)$ and let A^s be the inverse of the solution operator $S : L^2(\Omega) \rightarrow S(L^2(\Omega)) \subset X^s(\Omega)$, $f \mapsto u$ of the problem

$$\begin{cases} (-\Delta)^s u = f(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then A^s is a self-adjoint operator on $L^2(\Omega)$ and we have

$$\begin{aligned} (u, v) &= (A^{s/2}u, A^{s/2}v)_2 = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy, \quad \forall u, v \in X^s(\Omega), \\ \|u\| &= \|A^{s/2}u\|_2 = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}, \quad \forall u \in X^s(\Omega), \end{aligned}$$

where (\cdot, \cdot) and $(\cdot, \cdot)_2$ are the inner products in $X^s(\Omega)$ and $L^2(\Omega)$, respectively. Moreover, its spectrum $\sigma(A^s) \subset (0, \infty)$ and $(A^s)^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$ is a compact operator since the embedding $X^s(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Therefore $\sigma(A^s)$ consists of isolated eigenvalues λ_k , $k \geq 1$ of finite multiplicities satisfying $0 < \lambda_1 < \lambda_2 < \dots$. The first eigenvalue λ_1 is simple and has an associated eigenfunction $\varphi_1 > 0$, and if $w \in ((\mathbb{R}\varphi_1)^\perp \cap X^s(\Omega)) \setminus \{0\}$, then

$$0 = (w, \varphi_1) = (A^s w, \varphi_1)_2 = (w, A^s \varphi_1)_2 = \lambda_1 (w, \varphi_1)_2,$$

so $w^\pm \neq 0$. Hence the operator A^s satisfies all the assumptions of [41, Chapter 4].

Now we describe the minimal and maximal curves of $\Sigma^s(\Omega)$ in the square

$$Q_k = (\lambda_{k-1}, \lambda_{k+1})^2, \quad k \geq 2$$

constructed in [41]. Weak solutions of problem (3.1) coincide with critical points of the C^1 -functional

$$I(u, a, b) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \frac{1}{2} \int_{\Omega} [b(u^+)^2 + a(u^-)^2] dx, \quad u \in X^s(\Omega).$$

Denote by E_k the eigenspace of λ_k and set

$$N_k = \bigoplus_{j=1}^k E_j, \quad M_k = N_k^\perp \cap X^s(\Omega).$$

Then $X^s(\Omega) = N_k \oplus M_k$ is an orthogonal decomposition with respect to both (\cdot, \cdot) and $(\cdot, \cdot)_2$. When $(a, b) \in Q_k$, $I(v + y + w)$, $v + y + w \in N_{k-1} \oplus E_k \oplus M_k$ is strictly concave in v and strictly convex in w , i.e., if $v_1 \neq v_2 \in N_{k-1}$, $w \in M_{k-1}$,

$$I((1-t)v_1 + tv_2 + w) > (1-t)I(v_1 + w) + tI(v_2 + w) \quad \forall t \in (0, 1),$$

and if $v \in N_k$, $w_1 \neq w_2 \in M_k$,

$$I(v + (1-t)w_1 + tw_2) < (1-t)I(v + w_1) + tI(v + w_2) \quad \forall t \in (0, 1)$$

(see [41, Proposition 4.6.1]).

Proposition 3.1 ([41, Proposition 4.7.1, Corollary 4.7.3, Proposition 4.7.4]). *Let $(a, b) \in Q_k$.*

- (i) *There is a positive homogeneous map $\theta(\cdot, a, b) \in C(M_{k-1}, N_{k-1})$ such that $v = \theta(w)$ is the unique solution of*

$$I(v + w) = \sup_{v' \in N_{k-1}} I(v' + w), \quad w \in M_{k-1}.$$

Moreover, $I'(v + w) \perp N_{k-1}$ if and only if $v = \theta(w)$. Furthermore, θ is continuous on $M_{k-1} \times Q_k$ and $\theta(w, \lambda_k, \lambda_k) = 0$ for all $w \in M_{k-1}$.

- (ii) *There is a positive homogeneous map $\tau(\cdot, a, b) \in C(N_k, M_k)$ such that $w = \tau(v)$ is the unique solution of*

$$I(v + w) = \inf_{w' \in M_k} I(v + w'), \quad v \in N_k.$$

Moreover, $I'(v + w) \perp M_k$ if and only if $w = \tau(v)$. Furthermore, τ is continuous on $N_k \times Q_k$ and $\tau(v, \lambda_k, \lambda_k) = 0$ for all $v \in N_k$.

For $(a, b) \in Q_k$, let

$$\sigma(w, a, b) = \theta(w, a, b) + w, \quad w \in M_{k-1}, \quad S_k(a, b) = \sigma(M_{k-1}, a, b),$$

$$\zeta(v, a, b) = v + \tau(v, a, b), \quad v \in N_k, \quad S^k(a, b) = \zeta(N_k, a, b).$$

Then S_k and S^k are topological manifolds modeled on M_{k-1} and N_k , respectively. Thus, S_k is infinite dimensional, while S^k is d_k -dimensional, where $d_k = \dim N_k$. For $B \subset X^s(\Omega)$, set $\tilde{B} = \{u \in B : \|u\| = 1\}$. We say that B is a radial set if $B = \{tu : u \in \tilde{B}, t \geq 0\}$. Since θ and τ are positive homogeneous, so are σ and ζ , and hence S_k and S^k are radial manifolds.

Let

$$K(a, b) = \{u \in X^s(\Omega) : I'(u, a, b) = 0\}$$

be the set of critical points of $I(\cdot, a, b)$. Since I' is positive homogeneous, K is a radial set. Since $I(u) = (I'(u), u)/2$,

$$I(u) = 0 \quad \forall u \in K. \quad (3.2)$$

Since $X^s(\Omega) = N_{k-1} \oplus E_k \oplus M_k$, Proposition 3.1 implies

$$K = \{u \in S_k \cap S^k : I'(u) \perp E_k\}. \quad (3.3)$$

Together with (3.2), it also implies

$$K \subset \{u \in S_k \cap S^k : I(u) = 0\}. \quad (3.4)$$

Set

$$n_{k-1}(a, b) = \inf_{w \in \tilde{M}_{k-1}} \sup_{v \in N_{k-1}} I(v + w, a, b),$$

$$m_k(a, b) = \sup_{v \in \tilde{N}_k} \inf_{w \in \tilde{M}_k} I(v + w, a, b).$$

Since $I(u, a, b)$ is nonincreasing in a for fixed u and b , and in b for fixed u and a , $n_{k-1}(a, b)$ and $m_k(a, b)$ are nonincreasing in a for fixed b , and in b for fixed a . By Proposition 3.1,

$$n_{k-1}(a, b) = \inf_{w \in \tilde{M}_{k-1}} I(\sigma(w, a, b), a, b),$$

$$m_k(a, b) = \sup_{v \in \tilde{N}_k} I(\zeta(v, a, b), a, b).$$

Proposition 3.2 ([41, Proposition 4.7.5, Lemma 4.7.6, Proposition 4.7.7]). *Let $(a, b), (a', b') \in Q_k$.*

(i) *Assume that $n_{k-1}(a, b) = 0$. Then*

$$I(u, a, b) \geq 0 \quad \forall u \in S_k(a, b),$$

$$K(a, b) = \{u \in S_k(a, b) : I(u, a, b) = 0\},$$

and $(a, b) \in \Sigma^s(\Omega)$.

(a) *If $a' \leq a$, $b' \leq b$, and $(a', b') \neq (a, b)$, then $n_{k-1}(a', b') > 0$,*

$$I(u, a', b') > 0 \quad \forall u \in S_k(a', b') \setminus \{0\},$$

and $(a', b') \notin \Sigma^s(\Omega)$.

(b) *If $a' \geq a$, $b' \geq b$, and $(a', b') \neq (a, b)$, then $n_{k-1}(a', b') < 0$ and there is a $u \in S_k(a', b') \setminus \{0\}$ such that*

$$I(u, a', b') < 0.$$

Furthermore, n_{k-1} is continuous on Q_k and $n_{k-1}(\lambda_k, \lambda_k) = 0$.

(ii) *Assume that $m_k(a, b) = 0$. Then*

$$I(u, a, b) \leq 0 \quad \forall u \in S^k(a, b),$$

$$K(a, b) = \{u \in S^k(a, b) : I(u, a, b) = 0\},$$

and $(a, b) \in \Sigma^s(\Omega)$.

(a) *If $a' \geq a$, $b' \geq b$, and $(a', b') \neq (a, b)$, then $m_k(a', b') < 0$,*

$$I(u, a', b') < 0 \quad \forall u \in S^k(a', b') \setminus \{0\},$$

and $(a', b') \notin \Sigma^s(\Omega)$.

(b) *If $a' \leq a$, $b' \leq b$, and $(a', b') \neq (a, b)$, then $m_k(a', b') > 0$ and there is a $u \in S^k(a', b') \setminus \{0\}$ such that*

$$I(u, a', b') > 0.$$

Furthermore, m_k is continuous on Q_k and $m_k(\lambda_k, \lambda_k) = 0$.

For $a \in (\lambda_{k-1}, \lambda_{k+1})$, set

$$\nu_{k-1}(a) = \sup \{b \in (\lambda_{k-1}, \lambda_{k+1}) : n_{k-1}(a, b) \geq 0\},$$

$$\mu_k(a) = \inf \{b \in (\lambda_{k-1}, \lambda_{k+1}) : m_k(a, b) \leq 0\}.$$

Then

$$b = \nu_{k-1}(a) \iff n_{k-1}(a, b) = 0,$$

$$b = \mu_k(a) \iff m_k(a, b) = 0$$

(see [41, Lemma 4.7.8]).

Theorem 3.3 ([41, Theorem 4.7.9]). *Let $(a, b) \in Q_k$.*

(i) *The function ν_{k-1} is continuous, strictly decreasing, and satisfies*

- (a) $\nu_{k-1}(\lambda_k) = \lambda_k$,
- (b) $b = \nu_{k-1}(a) \implies (a, b) \in \Sigma^s(\Omega)$,
- (c) $b < \nu_{k-1}(a) \implies (a, b) \notin \Sigma^s(\Omega)$.

(ii) The function μ_k is continuous, strictly decreasing, and satisfies

- (a) $\mu_k(\lambda_k) = \lambda_k$,
- (b) $b = \mu_k(a) \implies (a, b) \in \Sigma^s(\Omega)$,
- (c) $b > \mu_k(a) \implies (a, b) \notin \Sigma^s(\Omega)$.

(iii) $\nu_{k-1}(a) \leq \mu_k(a)$.

Thus,

$$C_k : b = \nu_{k-1}(a), \quad C^k : b = \mu_k(a)$$

are strictly decreasing curves in Q_k that belong to $\Sigma^s(\Omega)$. They both pass through the point (λ_k, λ_k) and may coincide. The region

$$I_k = \{(a, b) \in Q_k : b < \nu_{k-1}(a)\}$$

below the lower curve C_k and the region

$$I^k = \{(a, b) \in Q_k : b > \mu_k(a)\}$$

above the upper curve C^k are free of $\Sigma^s(\Omega)$. They are the minimal and maximal curves of $\Sigma^s(\Omega)$ in Q_k in this sense. Points in the region

$$\Pi_k = \{(a, b) \in Q_k : \nu_{k-1}(a) < b < \mu_k(a)\}$$

between C_k and C^k , when it is nonempty, may or may not belong to $\Sigma^s(\Omega)$.

For $(a, b) \in Q_k$, let

$$\mathcal{N}_k(a, b) = S_k(a, b) \cap S^k(a, b).$$

Since S_k and S^k are radial sets, so is \mathcal{N}_k . The next two propositions show that \mathcal{N}_k is a topological manifold modeled on E_k and hence

$$\dim \mathcal{N}_k = d_k - d_{k-1}.$$

We will call it the null manifold of I .

Proposition 3.4 ([41, Proposition 4.8.1, Lemma 4.8.3, Proposition 4.8.4]). *Let $(a, b) \in Q_k$.*

- (i) *There is a positive homogeneous map $\eta(\cdot, a, b) \in C(E_k, N_{k-1})$ such that $v = \eta(y)$ is the unique solution of*

$$I(\zeta(v + y)) = \sup_{v' \in N_{k-1}} I(\zeta(v' + y)), \quad y \in E_k.$$

Moreover, $I'(\zeta(v + y)) \perp N_{k-1}$ if and only if $v = \eta(y)$. Furthermore, η is continuous on $E_k \times Q_k$ and $\eta(y, \lambda_k, \lambda_k) = 0$ for all $y \in E_k$.

- (ii) *There is a positive homogeneous map $\xi(\cdot, a, b) \in C(E_k, M_k)$ such that $w = \xi(y)$ is the unique solution of*

$$I(\sigma(y + w)) = \inf_{w' \in M_k} I(\sigma(y + w')), \quad y \in E_k.$$

Moreover, $I'(\sigma(y + w)) \perp M_k$ if and only if $w = \xi(y)$. Furthermore, ξ is continuous on $E_k \times Q_k$ and $\xi(y, \lambda_k, \lambda_k) = 0$ for all $y \in E_k$.

- (iii) *For all $y \in E_k$, $\zeta(\eta(y) + y) = \sigma(y + \xi(y))$, i.e., $\eta(y) = \theta(y + \xi(y))$ and $\xi(y) = \tau(\eta(y) + y)$.*

Let

$$\varphi(y) = \zeta(\eta(y) + y) = \sigma(y + \xi(y)), \quad y \in E_k.$$

Proposition 3.5 ([41, Proposition 4.8.5]). *Let $(a, b) \in Q_k$.*

(i) *$\varphi(\cdot, a, b) \in C(E_k, X^s(\Omega))$ is a positive homogeneous map such that*

$$I(\varphi(y)) = \inf_{w \in M_k} \sup_{v \in N_{k-1}} I(v + y + w) = \sup_{v \in N_{k-1}} \inf_{w \in M_k} I(v + y + w), \quad y \in E_k$$

and $I'(\varphi(y)) \in E_k$ for all $y \in E_k$.

(ii) *If $(a', b') \in Q_k$ with $a' \geq a$ and $b' \geq b$, then*

$$I(\varphi(y, a', b'), a', b') \leq I(\varphi(y, a, b), a, b) \quad \forall y \in E_k.$$

(iii) *φ is continuous on $E_k \times Q_k$.*

(iv) *$\varphi(y, \lambda_k, \lambda_k) = y \quad \forall y \in E_k$.*

(v) *$\mathcal{N}_k(a, b) = \{\varphi(y, a, b) : y \in E_k\}$.*

(vi) *$\mathcal{N}_k(\lambda_k, \lambda_k) = E_k$.*

By (3.3) and (3.4),

$$K = \{u \in \mathcal{N}_k : I'(u) \perp E_k\} \subset \{u \in \mathcal{N}_k : I(u) = 0\}. \quad (3.5)$$

The following theorem shows that the curves C_k and C^k are closely related to $\tilde{I} = I|_{\mathcal{N}_k}$.

Theorem 3.6 ([41, Theorem 4.8.6]). *Let $(a, b) \in Q_k$.*

(i) *If $b < \nu_{k-1}(a)$, then*

$$\tilde{I}(u, a, b) > 0 \quad \forall u \in \mathcal{N}_k(a, b) \setminus \{0\}.$$

(ii) *If $b = \nu_{k-1}(a)$, then*

$$\tilde{I}(u, a, b) \geq 0 \quad \forall u \in \mathcal{N}_k(a, b),$$

$$K(a, b) = \{u \in \mathcal{N}_k(a, b) : \tilde{I}(u, a, b) = 0\}.$$

(iii) *If $\nu_{k-1}(a) < b < \mu_k(a)$, then there are $u_i \in \mathcal{N}_k(a, b) \setminus \{0\}$, $i = 1, 2$ such that*

$$\tilde{I}(u_1, a, b) < 0 < \tilde{I}(u_2, a, b).$$

(iv) *If $b = \mu_k(a)$, then*

$$\tilde{I}(u, a, b) \leq 0 \quad \forall u \in \mathcal{N}_k(a, b),$$

$$K(a, b) = \{u \in \mathcal{N}_k(a, b) : \tilde{I}(u, a, b) = 0\}.$$

(v) *If $b > \mu_k(a)$, then*

$$\tilde{I}(u, a, b) < 0 \quad \forall u \in \mathcal{N}_k(a, b) \setminus \{0\}.$$

By (3.5), solutions of (3.1) are in \mathcal{N}_k . The set $K(a, b)$ of solutions is all of $\mathcal{N}_k(a, b)$ exactly when $(a, b) \in Q_k$ is on both C_k and C^k (see [41, Theorem 4.8.7]). When λ_k is a simple eigenvalue, \mathcal{N}_k is 1-dimensional and hence this implies that (a, b) is on exactly one of those curves if and only if

$$K(a, b) = \{t \varphi(y_0, a, b) : t \geq 0\}$$

for some $y_0 \in E_k \setminus \{0\}$ (see [41, Corollary 4.8.8]).

The following theorem gives a sufficient condition for the region Π_k to be nonempty.

Theorem 3.7 ([41, Theorem 4.9.1]). *If there is a $y \in E_k$ such that $|y^+|_2 \neq |y^-|_2$, then there is a neighborhood $N \subset Q_k$ of (λ_k, λ_k) such that every point $(a, b) \in N \setminus \{(\lambda_k, \lambda_k)\}$ with $a + b = 2\lambda_k$ is in Π_k .*

For the local problem (1.3), this result is due to Li et al. [28]. When λ_k is a simple eigenvalue, the region Π_k is free of $\Sigma^s(\Omega)$ (see [41, Theorem 4.10.1]). For problem (1.3), this is due to Gallouët and Kavian [22].

When $(a, b) \notin \Sigma^s(\Omega)$, 0 is the only critical point of I and its critical groups are given by

$$C_q(I, 0) = H_q(I^0, I^0 \setminus \{0\}), \quad q \geq 0,$$

where $I^0 = \{u \in X^s(\Omega) : I(u) \leq 0\}$ and H denotes singular homology. We take the coefficient group to be the field \mathbb{Z}_2 . The following theorem gives our main results on the critical groups.

Theorem 3.8 ([41, Theorem 4.11.2]). *Let $(a, b) \in Q_k \setminus \Sigma^s(\Omega)$.*

(i) *If $(a, b) \in I_k$, then*

$$C_q(I, 0) \approx \delta_{qd_{k-1}} \mathbb{Z}_2.$$

(ii) *If $(a, b) \in I^k$, then*

$$C_q(I, 0) \approx \delta_{qd_k} \mathbb{Z}_2.$$

(iii) *If $(a, b) \in \Pi_k$, then*

$$C_q(I, 0) = 0, \quad q \leq d_{k-1} \text{ or } q \geq d_k$$

and

$$C_q(I, 0) \approx \tilde{H}_{q-d_{k-1}-1}(\{u \in \mathcal{N}_k : I(u) < 0\}), \quad d_{k-1} < q < d_k,$$

where \tilde{H} denotes the reduced homology groups. In particular, $C_q(I, 0) = 0$ for all q when λ_k is simple.

For the local problem (1.3), this result is due to Dancer [12, 13] and Perera and Schechter [38, 39, 40]. It can be used, for example, to obtain nontrivial solutions of perturbed problems with nonlinearities that cross a curve of the Dancer-Fučík spectrum, via a comparison of the critical groups at zero and infinity. Consider the problem

$$\begin{cases} (-\Delta)^s u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (3.6)$$

where f is a Carathéodory function on $\Omega \times \mathbb{R}$.

Theorem 3.9 (see [41, Theorem 5.6.1]). *If*

$$f(x, t) = \begin{cases} b_0 t^+ - a_0 t^- + o(t) & \text{as } t \rightarrow 0 \\ bt^+ - at^- + o(t) & \text{as } |t| \rightarrow \infty, \end{cases}$$

uniformly a.e. in Ω , for some (a_0, b_0) and (a, b) in $Q_k \setminus \Sigma^s(\Omega)$ that are on opposite sides of C_k or C^k , then problem (3.6) has a nontrivial weak solution.

For problem (1.3), this was proved in Perera and Schechter [39]. It generalizes a well-known result of Amann and Zehnder [1] on the existence of nontrivial solutions for problems crossing an eigenvalue.

References

- [1] H. Amann and E. Zehnder. Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 7(4):539–603, 1980. [10](#)
- [2] A. Anane and N. Tsouli. On the second eigenvalue of the p -Laplacian. In *Nonlinear partial differential equations (Fès, 1994)*, volume 343 of *Pitman Res. Notes Math. Ser.*, pages 1–9. Longman, Harlow, 1996. [3](#)
- [3] L. Brasco and E. Parini. The second eigenvalue of the fractional p -Laplacian. Preprint. [3](#)
- [4] A. Anane. Simplicité et isolation de la première valeur propre du p -laplacien avec poids. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(16):725–728, 1987. [3](#)
- [5] N.P. Căc. On nontrivial solutions of a Dirichlet problem whose jumping nonlinearity crosses a multiple eigenvalue. *J. Differential Equations*, 80(2):379–404, 1989. [3](#)
- [6] L. Caffarelli. Non-local diffusions, drifts and games. In *Nonlinear Partial Differential Equations*, volume 7 of *Abel Symposia*, pages 37–52, 2012. [2](#)
- [7] M. Cuesta. On the Fučík spectrum of the Laplacian and the p -Laplacian. In *Proceedings of Seminar in Differential Equations*, pages 67–96, Kvilda, Czech Republic, May 29 – June 2, 2000. Centre of Applied Mathematics, Faculty of Applied Sciences, University of West Bohemia in Pilsen. [3](#)
- [8] M. Cuesta, D. de Figueiredo, and J.-P. Gossez. The beginning of the Fučík spectrum for the p -Laplacian. *J. Differential Equations*, 159(1):212–238, 1999. [3](#)
- [9] M. Cuesta and J.-P. Gossez. A variational approach to nonresonance with respect to the Fučík spectrum. *Nonlinear Anal.*, 19(5):487–500, 1992. [3](#)
- [10] E.N. Dancer. On the Dirichlet problem for weakly non-linear elliptic partial differential equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 76(4):283–300, 1976/77. [2](#)
- [11] E.N. Dancer. Corrigendum: “On the Dirichlet problem for weakly nonlinear elliptic partial differential equations” [Proc. Roy. Soc. Edinburgh Sect. A **76** (1976/77), no. 4, 283–300; MR **58** #17506]. *Proc. Roy. Soc. Edinburgh Sect. A*, 89(1-2):15, 1981. [2](#)
- [12] E.N. Dancer. Remarks on jumping nonlinearities. In *Topics in nonlinear analysis*, volume 35 of *Progr. Nonlinear Differential Equations Appl.*, pages 101–116. Birkhäuser, Basel, 1999. [10](#)
- [13] E.N. Dancer. Some results for jumping nonlinearities. *Topol. Methods Nonlinear Anal.*, 19(2):221–235, 2002. [10](#)
- [14] D.G. de Figueiredo and J.-P. Gossez. On the first curve of the Fučík spectrum of an elliptic operator. *Differential Integral Equations*, 7(5-6):1285–1302, 1994. [3](#)
- [15] A. Di Castro, T. Kuusi, and G. Palatucci. Local behavior of fractional p -minimizers. preprint. [2](#)
- [16] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012. [2](#)
- [17] P. Drábek. *Solvability and bifurcations of nonlinear equations*, volume 264 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1992. [3](#)
- [18] P. Drábek and S.B. Robinson. Resonance problems for the p -Laplacian. *J. Funct. Anal.*, 169(1):189–200, 1999. [3](#)

- [19] E.R. Fadell and P.H. Rabinowitz. Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. *Invent. Math.*, 45(2):139–174, 1978. [4](#)
- [20] G. Franzina and G. Palatucci. Fractional p -eigenvalues. *Riv. Mat. Univ. Parma*, to appear. [2](#), [4](#)
- [21] S. Fučík. Boundary value problems with jumping nonlinearities. *Časopis Pěst. Mat.*, 101(1):69–87, 1976. [2](#)
- [22] T. Gallouët and O. Kavian. Résultats d’existence et de non-existence pour certains problèmes demi-linéaires à l’infini. *Ann. Fac. Sci. Toulouse Math. (5)*, 3(3-4):201–246 (1982), 1981. [3](#), [10](#)
- [23] Sarika Goyal and K. Sreenadh. On the Fučík spectrum of non-local elliptic operators. *NoDEA Nonlinear Differential Equations Appl.*, 21(4):567–588, 2014. [3](#)
- [24] A. Iannizzotto, S. Liu, K. Perera, and M. Squassina. Existence results for fractional p -Laplacian problems via Morse theory. preprint. [2](#)
- [25] A. Iannizzotto and M. Squassina. Weyl-type laws for fractional p -eigenvalue problems. *Asymptot. Anal.*, 88(4):233–245, 2014. [2](#)
- [26] A.C. Lazer and P.J. McKenna. Critical point theory and boundary value problems with nonlinearities crossing multiple eigenvalues. II. *Comm. Partial Differential Equations*, 11(15):1653–1676, 1986. [3](#)
- [27] A. Lazer. Introduction to multiplicity theory for boundary value problems with asymmetric nonlinearities. In *Partial differential equations (Rio de Janeiro, 1986)*, volume 1324 of *Lecture Notes in Math.*, pages 137–165. Springer, Berlin, 1988. [3](#)
- [28] C. Li, S. Li, and Z. Liu. Existence of type (II) regions and convexity and concavity of potential functionals corresponding to jumping nonlinear problems. *Calc. Var. Partial Differential Equations*, 32(2):237–251, 2008. [10](#)
- [29] E. Lindgren and P. Lindqvist. Fractional eigenvalues. *Calc. Var. Partial Differential Equations*, 49(1-2):795–826, 2014. [2](#), [4](#)
- [30] P. Lindqvist. On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. *Proc. Amer. Math. Soc.*, 109(1):157–164, 1990. [3](#)
- [31] P. Lindqvist. Addendum: “On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$ ” [Proc. Amer. Math. Soc. 109 (1990), no. 1, 157–164; MR 90h:35088]. *Proc. Amer. Math. Soc.*, 116(2):583–584, 1992. [3](#)
- [32] C.A. Magalhães. Semilinear elliptic problem with crossing of multiple eigenvalues. *Comm. Partial Differential Equations*, 15(9):1265–1292, 1990. [3](#)
- [33] C.A. Margulies and W. Margulies. An example of the Fučík spectrum. *Nonlinear Anal.*, 29(12):1373–1378, 1997. [3](#)
- [34] A.M. Micheletti and A. Pistoia. On the Fučík spectrum for the p -Laplacian. *Differential Integral Equations*, 14(7):867–882, 2001. [3](#)
- [35] K. Perera. Nontrivial critical groups in p -Laplacian problems via the Yang index. *Topol. Methods Nonlinear Anal.*, 21(2):301–309, 2003. [3](#)
- [36] K. Perera. On the Fučík spectrum of the p -Laplacian. *NoDEA Nonlinear Differential Equations Appl.*, 11(2):259–270, 2004. [3](#)

- [37] K. Perera, R.P. Agarwal, and D. O'Regan. *Morse theoretic aspects of p -Laplacian type operators*, volume 161 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010. [3](#), [4](#)
- [38] K. Perera and M. Schechter. Type II regions between curves of the Fučík spectrum and critical groups. *Topol. Methods Nonlinear Anal.*, 12(2):227–243, 1998. [10](#)
- [39] K. Perera and M. Schechter. A generalization of the Amann-Zehnder theorem to nonresonance problems with jumping nonlinearities. *NoDEA Nonlinear Differential Equations Appl.*, 7(4):361–367, 2000. [10](#)
- [40] K. Perera and M. Schechter. The Fučík spectrum and critical groups. *Proc. Amer. Math. Soc.*, 129(8):2301–2308 (electronic), 2001. [10](#)
- [41] K. Perera and M. Schechter. *Topics in critical point theory*, volume 198 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. [3](#), [5](#), [6](#), [7](#), [8](#), [9](#), [10](#)
- [42] B. Ruf. On nonlinear elliptic problems with jumping nonlinearities. *Ann. Mat. Pura Appl. (4)*, 128:133–151, 1981. [3](#)
- [43] M. Schechter. The Fučík spectrum. *Indiana Univ. Math. J.*, 43(4):1139–1157, 1994. [3](#)